

Review, Identities, Theorems, Formulas and Tables for Math 3710

Let $n, \bar{n}, m, \bar{m}, k, \bar{k}, l, p, q$ and \bar{q} be nonnegative integers, unless stated otherwise. Let z be a nonnegative real number, unless stated otherwise.

Trigonometric Identities

$$1. \sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)] \quad 2. \sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$$

$$3. \cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

Hyperbolic Functions

$$4. \sinh x = \frac{e^x - e^{-x}}{2} \quad 5. \cosh x = \frac{e^x + e^{-x}}{2} \quad 6. \tanh x = \frac{\sinh x}{\cosh x} \quad 7. \coth x = \frac{\cosh x}{\sinh x} \quad 8. \operatorname{sech} x = \frac{1}{\cosh x}$$

$$9. \operatorname{csch} x = \frac{1}{\sinh x}$$

$$10. \sinh(-x) = -\sinh x \quad 11. \cosh(-x) = \cosh x \quad 12. \cosh^2 x - \sinh^2 x = 1 \quad 13. 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$14. \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \quad 15. \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$16. \frac{d}{dx}(\sinh x) = \cosh x \quad 17. \frac{d}{dx}(\cosh x) = \sinh x \quad 18. \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad 19. \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$20. \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \quad 21. \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Integrals

$$22. \int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C \quad 23. \int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$$

$$24. \int x^2 \sin ax \, dx = \frac{2}{a^3} \cos ax + \frac{2}{a^2} x \sin ax - \frac{1}{a} x^2 \cos ax + C$$

$$25. \int x^2 \cos ax \, dx = -\frac{2}{a^3} \sin ax + \frac{2}{a^2} x \cos ax + \frac{1}{a} x^2 \sin ax + C$$

$$26. \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C \quad 27. \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

Definite Integrals

$$28. \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad 29. \int_0^a \cos \frac{n\pi x}{a} \cos \frac{m\pi x}{a} \, dx = \begin{cases} a, & \text{if } n = m = 0 \\ \frac{a}{2}, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$30. \text{For positive integers } n \text{ and } m, \int_0^a \cos \frac{(2n-1)\pi x}{2a} \cos \frac{(2m-1)\pi x}{2a} \, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$$

$$31. \text{For positive integers } n \text{ and } m, \int_0^a \sin \frac{(2n-1)\pi x}{2a} \sin \frac{(2m-1)\pi x}{2a} \, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$$

$$32. \int_{-a}^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx = \begin{cases} a, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad 33. \int_{-a}^a \cos \frac{n\pi x}{a} \cos \frac{m\pi x}{a} \, dx = \begin{cases} 2a, & \text{if } n = m = 0 \\ a, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$34. \int_{-a}^a \sin \frac{n\pi x}{a} \cos \frac{m\pi x}{a} \, dx = 0$$

35. For $0 < \alpha_1 < \alpha_2 < \dots$ zeros of $J_z(x)$ and $z \geq 0$, $\int_0^a J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\bar{m}} r}{a}) r dr = \begin{cases} \frac{a^2}{2} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \\ 0, & \text{otherwise} \end{cases}$

36. For $0 < \beta_1 < \beta_2 < \dots$ zeros of $J'_0(x)$, $\int_0^a J_0(\frac{\beta_m r}{a}) r dr = 0$ and

$$\int_0^a J_0(\frac{\beta_m r}{a}) J_0(\frac{\beta_{\bar{m}} r}{a}) r dr = \begin{cases} \frac{a^2}{2} J_0^2(\beta_m), & \text{if } \bar{m} = m \\ 0, & \text{otherwise} \end{cases}$$

37. For $0 < \beta_1 < \beta_2 < \dots$ zeros of $J'_z(x)$ and $z > 0$,

$$\int_0^a J_z(\frac{\beta_m r}{a}) J_z(\frac{\beta_{\bar{m}} r}{a}) r dr = \begin{cases} \frac{a^2}{2} [J_z^2(\beta_m) - J_{z-1}(\beta_m) J_{z+1}(\beta_m)], & \text{if } \bar{m} = m \\ 0, & \text{otherwise} \end{cases}$$

38*. $\int_0^\pi P_n^m(\cos \phi) P_{\bar{n}}^m(\cos \phi) \sin \phi d\phi = \begin{cases} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$

39*. $\int_{-1}^1 P_n^m(s) P_{\bar{n}}^m(s) ds = \begin{cases} \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$

* For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Double Integrals

40. $\int_0^a \int_0^b \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} dy dx = \begin{cases} \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$

41. $\int_0^a \int_0^b \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \sin \frac{p\pi x}{a} \cos \frac{q\pi y}{b} dy dx = \begin{cases} \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$

42. $\int_0^a \int_0^b \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{b} dy dx = \begin{cases} ab, & \text{if } n = m = p = q = 0 \\ \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{2}, & \text{if } n = p = 0 \text{ and } m = q \neq 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

43. $\int_0^a \int_0^b J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\bar{m}} r}{a}) r \cos \frac{k\pi \theta}{b} \cos \frac{\bar{k}\pi \theta}{b} d\theta dr = \begin{cases} \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k = 0 \\ \frac{a^2 b}{4} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$

44. $\int_0^a \int_0^b J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\bar{m}} r}{a}) r \sin \frac{k\pi \theta}{b} \sin \frac{\bar{k}\pi \theta}{b} d\theta dr = \begin{cases} \frac{a^2 b}{4} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$

45. $\int_0^a \int_{-b}^b J_z(\frac{\alpha_m r}{a}) J_z(\frac{\alpha_{\bar{m}} r}{a}) r \cos \frac{k\pi \theta}{b} \cos \frac{\bar{k}\pi \theta}{b} d\theta dr = \begin{cases} a^2 b J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k = 0 \\ \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

$$46. \int_0^a \int_{-b}^b J_z\left(\frac{\alpha_m r}{a}\right) J_z\left(\frac{\alpha_{\bar{m}} r}{L}\right) r \sin \frac{k\pi\theta}{b} \sin \frac{\bar{k}\pi\theta}{b} d\theta dr = \begin{cases} \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$47^*. \int_0^\pi \int_0^b P_n^m(\cos \phi) P_{\bar{n}}^{\bar{m}}(\cos \phi) \sin \phi \cos \frac{k\pi\theta}{b} \cos \frac{\bar{k}\pi\theta}{b} d\theta d\phi = \begin{cases} \frac{2b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k = 0 \\ \frac{b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$48^*. \int_0^\pi \int_0^b P_n^m(\cos \phi) P_{\bar{n}}^{\bar{m}}(\cos \phi) \sin \phi \sin \frac{k\pi\theta}{b} \sin \frac{\bar{k}\pi\theta}{b} d\theta d\phi = \begin{cases} \frac{b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$49^*. \int_0^\pi \int_{-b}^b P_n^m(\cos \phi) P_{\bar{n}}^{\bar{m}}(\cos \phi) \sin \phi \cos \frac{k\pi\theta}{b} \cos \frac{\bar{k}\pi\theta}{b} d\theta d\phi = \begin{cases} \frac{4b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k = 0 \\ \frac{2b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$50^*. \int_0^\pi \int_{-b}^b P_n^m(\cos \phi) P_{\bar{n}}^{\bar{m}}(\cos \phi) \sin \phi \sin \frac{k\pi\theta}{b} \sin \frac{\bar{k}\pi\theta}{b} d\theta d\phi = \begin{cases} \frac{2b}{2n+1} \frac{(n+m)!}{(n-m)!}, & \text{if } \bar{n} = n \geq m \text{ and } \bar{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

* It also holds if $k = m$. For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Triple Integrals

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

$$51^{**}. \int_0^a \int_0^\pi \int_0^b J_z\left(\frac{\alpha_m \rho}{a}\right) J_z\left(\frac{\alpha_{\bar{m}} \rho}{a}\right) \rho P_k^l(\cos \phi) P_{\bar{k}}^l(\cos \phi) \sin \phi \cos \frac{q\pi\theta}{b} \cos \frac{\bar{q}\pi\theta}{b} d\theta d\phi d\rho = \begin{cases} \frac{a^2 b}{2k+1} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q = 0 \\ \frac{a^2 b}{2(2k+1)} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$52^{**}. \int_0^a \int_0^\pi \int_0^b J_z\left(\frac{\alpha_m \rho}{a}\right) J_z\left(\frac{\alpha_{\bar{m}} \rho}{a}\right) \rho P_k^l(\cos \phi) P_{\bar{k}}^l(\cos \phi) \sin \phi \sin \frac{q\pi\theta}{b} \sin \frac{\bar{q}\pi\theta}{b} d\theta d\phi d\rho = \begin{cases} \frac{a^2 b}{2(2k+1)} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$53^{**}. \int_0^a \int_0^\pi \int_{-b}^b J_z\left(\frac{\alpha_m \rho}{a}\right) J_z\left(\frac{\alpha_{\bar{m}} \rho}{a}\right) \rho P_k^l(\cos \phi) P_{\bar{k}}^l(\cos \phi) \sin \phi \cos \frac{q\pi\theta}{b} \cos \frac{\bar{q}\pi\theta}{b} d\theta d\phi d\rho = \begin{cases} \frac{2a^2 b}{2k+1} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q = 0 \\ \frac{a^2 b}{2k+1} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$54^{**}. \int_0^a \int_0^\pi \int_{-b}^b J_z\left(\frac{\alpha_m \rho}{a}\right) J_z\left(\frac{\alpha_{\bar{m}} \rho}{a}\right) \rho P_k^l(\cos \phi) P_{\bar{k}}^l(\cos \phi) \sin \phi \sin \frac{q\pi\theta}{b} \sin \frac{\bar{q}\pi\theta}{b} d\theta d\phi d\rho = \begin{cases} \frac{a^2 b}{2k+1} \frac{(k+l)!}{(k-l)!} J_{z+1}^2(\alpha_m), & \text{if } \bar{m} = m, \bar{k} = k \geq l \text{ and } \bar{q} = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

** For $l = 0$, $P_n^l(s) = P_n(s)$ and $\frac{(k+l)!}{(k-l)!} = 1$.

Ordinary Differential Equations

55. First Order Linear ODE: $y' + f(x)y = g(x)$

Integrating Factor: $\mu(x) = e^{\int f(x)dx}$ with $C = 0$, $\mu(x)y'(x) + \mu(x)f(x)y(x) = \mu(x)g(x) \implies$

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)g(x) \implies \mu(x)y(x) = \int \mu(x)g(x) dx + C \implies y(x) = \frac{1}{\mu(x)} \int \mu(x)g(x) dt + \frac{C}{\mu(x)}$$

Or, integrating factor: $\mu(x) = e^{\int_{x_0}^x f(t)dt}$ and $y(x) = \frac{1}{\mu(x)} \int_{x_0}^x \mu(t)g(t) dt + \frac{y(x_0)}{\mu(x)}$

56. First Order Separable ODE: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Implicit Solution: $\int h(y) dy = \int g(x) dx \implies b(y) = G(x) + C$ with $b' = h$ and $G' = g$

Or, $\int_{y(x_0)}^y h(t) dt = \int_{x_0}^x g(t) dt$

57. Second Order Linear ODE with Constant Coefficients: $ay'' + by' + cy = 0$

Characteristics Equation: $ar^2 + br + c = 0$ with solutions r_1 and r_2

$$y(x) = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x}, & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and unequal} \\ c_1 e^{r_1 x} + c_2 x e^{r_1 x}, & \text{if } r_1 = r_2 \\ c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x, & \text{if } r_1, r_2 = \lambda \pm \mu i \end{cases}$$

If $r_1, r_2 = \pm r$, then $y(x) = c_1 e^{-rx} + c_2 e^{rx}$ or $y(x) = c_1 \cosh rx + c_2 \sinh rx$ or

$$y(x) = c_1 \cosh r(x - x_0) + c_2 \sinh r(x - x_0) \text{ or}$$

$$y(x) = c_1 \sinh r(x - x_0) + c_2 \cosh r(x - x_0) \text{ or } y(x) = c_1 \cosh r(x - x_0) + c_2 \sinh r(x - x_0)$$

58. Second Order Linear Nonhomogeneous ODE: $y'' + p(x)y' + q(x)y = g(x)$

General Solution: $y(x) = y_h(x) + y_p(x)$ where the homogeneous solution $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, while y_1 and y_2 are two linearly independent solutions of the same homogeneous equation, and the particular solution $y_p(x)$ is a solution to the nonhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$.

Method of Variation of Parameter: $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $u_1'(x) = \frac{-y_2(x)g(x)}{W(x)}$,

$u_2'(x) = \frac{y_1(x)g(x)}{W(x)}$ and the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

$$y_p(x) = y_1(x) \int \frac{-y_2(x)g(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)g(x)}{W(x)} dx \text{ or}$$

$$y_p(x) = y_1(x) \int_{x_0}^x \frac{-y_2(t)g(t)}{W(t)} dt + y_2(x) \int_{x_0}^x \frac{y_1(t)g(t)}{W(t)} dt$$

59. Cauchy-Euler Equation: $x^2 y'' + \alpha x y' + \beta y = 0$

Indicial Equation: $p(p-1) + \alpha p + \beta = 0$ with solutions p_1 and p_2

$$y(x) = \begin{cases} c_1 |x|^{p_1} + c_2 |x|^{p_2}, & \text{if } p_1 \text{ and } p_2 \text{ are real-valued and unequal} \\ (c_1 + c_2 \ln |x|) |x|^{p_1}, & \text{if } p_1 = p_2 \\ |x|^\lambda [c_1 \cos(\mu \ln |x|) + c_2 \sin(\mu \ln |x|)], & \text{if } p_1, p_2 = \lambda \pm \mu i \end{cases}$$

60. $x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} - n^2 \phi = 0$ and $\phi(0)$ bounded $\implies \phi(x) = x^n$ for $n = 0, 1, \dots$

Rayleigh Quotients

$$61. \frac{d}{dx} \left[s(x) \frac{d\phi}{dx} \right] - q(x)\phi + \lambda p(x)\phi = 0 \implies \lambda = \frac{-s(x)\phi(x) \frac{d\phi}{dx} \Big|_l^r + \int_l^r \left[s(x) \left(\frac{d\phi}{dx} \right)^2 + q(x)\phi^2(x) \right] dx}{\int_l^r \phi^2(x)p(x) dx}$$

$$62. \nabla^2 \phi + \lambda \phi = 0 \implies \lambda = \frac{-\oint_C \phi \nabla \phi \cdot \hat{n} ds + \iint_R |\nabla \phi|^2 dA}{\iint_R \phi^2 dA}$$

Lagrange's Identity and Green's Formula

For $L(\phi) = \frac{d}{dx} \left[s(x) \frac{d\phi}{dx} \right] - q(x)\phi$,

$$63. uL(v) - vL(u) = \frac{d}{dx} \left[s(x) \left(u(x) \frac{dv}{dx} - v(x) \frac{du}{dx} \right) \right]$$

$$64. \int_l^r [uL(v) - vL(u)] dx = s(x) \left[u(x) \frac{dv}{dx} - v(x) \frac{du}{dx} \right] \Big|_l^r$$

Green's Identities

$$65. \iint_R u \nabla^2 v dA = \oint_C u \nabla v \cdot \hat{n} ds - \iint_R \nabla u \cdot \nabla v dA$$

$$66. \iint_R (u \nabla^2 v - v \nabla^2 u) dA = \oint_C (u \nabla v - v \nabla u) \cdot \hat{n} ds$$

$$67. \iiint_{\Omega} (u \nabla^2 v - v \nabla^2 u) dV = \iiint_{\partial \Omega} (u \nabla v - v \nabla u) \cdot \hat{n} dS$$

Eigenvalue Problems

$$68. \frac{d^2 \phi}{dx^2} = \lambda \phi, \phi(0) = 0 \text{ and } \phi(a) = 0 \implies \lambda = -\left(\frac{n\pi}{a}\right)^2, \phi(x) = \sin \frac{n\pi x}{a} \text{ for } n = 1, 2, \dots$$

$$69. \frac{d^2 \phi}{dx^2} = \lambda \phi, \frac{d\phi}{dx}(0) = 0 \text{ and } \frac{d\phi}{dx}(a) = 0 \implies \lambda = -\left(\frac{n\pi}{a}\right)^2, \phi(x) = \cos \frac{n\pi x}{a} \text{ for } n = 0, 1, \dots$$

$$70. \begin{cases} \frac{d^2 \phi}{dx^2} = \lambda \phi \\ \phi(-a) = \phi(a) \\ \frac{d\phi}{dx}(-a) = \frac{d\phi}{dx}(a) \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 \\ \phi(x) = \cos \frac{n\pi x}{a} \text{ and } \sin \frac{n\pi x}{a} \end{cases} \text{ for } n = 0, 1, \dots$$

$$71. \begin{cases} \frac{d^2 \phi}{dx^2} = \lambda \phi \\ \phi(0) = 0 \\ \frac{d\phi}{dx}(a) = 0 \end{cases} \implies \begin{cases} \lambda = -\left[\frac{(2n-1)\pi}{2a}\right]^2 \\ \phi(x) = \sin \frac{(2n-1)\pi x}{2a} \end{cases} \text{ for } n = 1, 2, \dots$$

$$72. \begin{cases} \frac{d^2\phi}{dx^2} = \lambda\phi \\ \frac{d\phi}{dx}(0) = 0 \\ \phi(a) = 0 \end{cases} \implies \begin{cases} \lambda = -\left[\frac{(2n-1)\pi}{2a}\right]^2 \\ \phi(x) = \cos \frac{(2n-1)\pi x}{2a} \end{cases} \text{ for } n = 1, 2, \dots$$

$$73. \begin{cases} x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} + (-\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \phi(a) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{\alpha_m}{a}\right)^2 \\ \phi(x) = J_n\left(\frac{\alpha_m x}{a}\right) \end{cases} \text{ for } 0 < \alpha_1 < \alpha_2 < \dots \text{ zeros of } J_n(x)$$

$$74. \begin{cases} \frac{d}{d\rho} \left[\rho^2 \frac{df}{d\rho} \right] + [-\lambda\rho^2 - n(n+1)]f(\rho) = 0 \\ f(0) \text{ bounded} \\ f(a) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{\alpha_k}{a}\right)^2 \\ f(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}\left(\frac{\alpha_k \rho}{a}\right) \end{cases} \text{ for } 0 < \alpha_1 < \alpha_2 < \dots \text{ zeros of } J_{n+\frac{1}{2}}(\rho)$$

$$75. \begin{cases} x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} + (-\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \frac{d\phi}{dx}(a) = 0 \end{cases} \implies \begin{cases} n = 0, \lambda = 0, \phi(x) = 1 \\ n > 0, \lambda = -\left(\frac{\beta_m}{a}\right)^2, \phi(x) = J_n\left(\frac{\beta_m x}{a}\right) \end{cases} \text{ for } 0 < \beta_1 < \beta_2 < \dots \text{ zeros of } J'_n(x)$$

$$76^*. \begin{cases} \frac{d}{d\phi} \left[\sin\phi \frac{dg}{d\phi} \right] + \left(-\mu \sin\phi - \frac{m^2}{\sin\phi} \right) g(\phi) = 0 \\ g(0) \text{ and } g(\pi) \text{ bounded} \end{cases} \implies \begin{cases} \mu = -n(n+1) \\ g(\phi) = P_n^m(\cos\phi) \end{cases} \text{ for } n = m, m+1, \dots$$

$$77^*. \begin{cases} (1-s^2) \frac{d^2\phi}{ds^2} - 2s \frac{d\phi}{ds} + \left(-\mu - \frac{m^2}{1-s^2} \right) \phi = 0 \\ \phi(-1) \text{ and } \phi(1) \text{ bounded} \end{cases} \implies \begin{cases} \mu = -n(n+1) \\ \phi(s) = P_n^m(s) \end{cases} \text{ for } n = m, m+1, \dots$$

* For $m = 0$, $P_n^m(s) = P_n(s)$.

Two-Dimensional Eigenvalue Problems

$$78. \begin{cases} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \lambda\phi(x, y) \\ \phi(0, y) = \phi(a, y) = 0 \\ \phi(x, 0) = \phi(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 \\ \phi(x) = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

$$79. \begin{cases} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \lambda\phi(x, y) \\ \frac{\partial\phi}{\partial x}(0, y) = \frac{\partial\phi}{\partial x}(a, y) = 0 \\ \frac{\partial\phi}{\partial y}(x, 0) = \frac{\partial\phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 \\ \phi(x) = \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \end{cases} \text{ for } n = 0, 1, \dots \text{ and } m = 0, 1, \dots$$

$$80. \begin{cases} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \lambda\phi(x, y) \\ \phi(0, y) = \phi(a, y) = 0 \\ \frac{\partial\phi}{\partial y}(x, 0) = \frac{\partial\phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 \\ \phi(x) = \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 0, 1, \dots$$

$$81. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \frac{\partial \phi}{\partial x}(0, y) = \frac{\partial \phi}{\partial x}(a, 0) = 0 \\ \phi(x, 0) = \phi(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 \\ \phi(x) = \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \end{cases} \text{ for } n = 0, 1, \dots \text{ and } m = 1, 2, \dots$$

$$82. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \phi(0, y) = \phi(a, y) = 0 \\ \frac{\partial \phi}{\partial y}(x, 0) = \phi(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \sin \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

$$83. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \phi(0, y) = \phi(a, y) = 0 \\ \phi(x, 0) = \frac{\partial \phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \sin \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

$$84. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \frac{\partial \phi}{\partial x}(0, y) = \frac{\partial \phi}{\partial x}(a, y) = 0 \\ \frac{\partial \phi}{\partial y}(x, 0) = \phi(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \cos \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 0, 1, \dots \text{ and } m = 1, 2, \dots$$

$$85. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \frac{\partial \phi}{\partial x}(0, y) = \frac{\partial \phi}{\partial x}(a, y) = 0 \\ \phi(x, 0) = \frac{\partial \phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left(\frac{n\pi}{a}\right)^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \cos \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 0, 1, \dots \text{ and } m = 1, 2, \dots$$

$$86. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \frac{\partial \phi}{\partial x}(0, y) = \phi(a, y) = 0 \\ \phi(x, 0) = \frac{\partial \phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left[\frac{(2n-1)\pi}{2a}\right]^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \cos \frac{(2n-1)\pi x}{2a} \sin \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

$$87. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \frac{\partial \phi}{\partial x}(0, y) = \phi(a, y) = 0 \\ \frac{\partial \phi}{\partial y}(x, 0) = \phi(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left[\frac{(2n-1)\pi}{2a}\right]^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \cos \frac{(2n-1)\pi x}{2a} \cos \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

$$88. \begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda \phi(x, y) \\ \phi(0, y) = \frac{\partial \phi}{\partial x}(a, y) = 0 \\ \phi(x, 0) = \frac{\partial \phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{cases} \lambda = -\left[\frac{(2n-1)\pi}{2a}\right]^2 - \left[\frac{(2m-1)\pi}{2b}\right]^2 \\ \phi(x) = \sin \frac{(2n-1)\pi x}{2a} \sin \frac{(2m-1)\pi y}{2b} \end{cases} \text{ for } n = 1, 2, \dots \text{ and } m = 1, 2, \dots$$

Supporting Theorems

89. Green's Theorem (vector version)

Let R be a region in \mathfrak{R}^2 bounded by a piecewise-smooth, simple closed curve C with counterclockwise orientation. Let \vec{F} be a vector field whose components have continuous partial derivatives on an open region containing R , then
$$\iint_R \nabla \cdot \vec{F} \, dA = \oint_C \vec{F} \cdot \hat{n} \, ds.$$

90. Divergence Theorem

Let Ω be a simple solid region in \mathfrak{R}^3 and let $\partial\Omega$ be its boundary with the outward orientation. Let \vec{F} be a vector field whose components have continuous partial derivatives on an open region containing Ω , then
$$\iiint_{\Omega} \nabla \cdot \vec{F} \, dV = \iint_{\partial\Omega} \vec{F} \cdot \hat{n} \, dS.$$

91. If function f is continuous, $f(x) \not\equiv 0$ and $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx > 0$.

92. For a continuous nonnegative function f if $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ for $a \leq x \leq b$.

93. Uniform Convergence Definition

The sequence of functions $f_n : D \rightarrow \mathfrak{R}$, $n = 1, 2, \dots$, is said to converge uniformly to the function $f : D \rightarrow \mathfrak{R}$ if for every $\epsilon > 0$, there is a natural number N such that for all $x \in D$ we have $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

94. Weierstrass M Test (A test for uniform convergence.)

Suppose for each function $f_n : D \rightarrow \mathfrak{R}$, $n = 1, 2, \dots$, there exists a constant M_n with $|f_n(x)| \leq M_n$ for all $x \in D$, and $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

95. Interchanging Limit and Integral

Suppose functions $f_n : [a, b] \rightarrow \mathfrak{R}$, $n = 1, 2, \dots$, are continuous and converge uniformly to a function $f : [a, b] \rightarrow \mathfrak{R}$. Then
$$\lim_{n \rightarrow \infty} \left[\int_a^b f_n(x) \, dx \right] = \int_a^b \left[\lim_{n \rightarrow \infty} f_n(x) \right] \, dx = \int_a^b f(x) \, dx.$$

96. Interchanging Integral and Summation

Suppose functions $f_n : [a, b] \rightarrow \mathfrak{R}$, $n = 1, 2, \dots$, are continuous, and $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Then
$$\int_a^b \left[\sum_{n=1}^{\infty} f_n(x) \right] \, dx = \sum_{n=1}^{\infty} \left[\int_a^b f_n(x) \, dx \right].$$

97. Interchanging Differentiation and Summation

Suppose functions f_n , $n = 1, 2, \dots$, are continuously differentiable, $\sum_{n=1}^{\infty} f_n$ converges pointwise,

and $\sum_{n=1}^{\infty} f'_n$ converges uniformly. Then
$$\frac{d}{dx} \left[\sum_{n=1}^{\infty} f_n(x) \right] = \sum_{n=1}^{\infty} \left[\frac{d}{dx} f_n(x) \right].$$

98. Leibniz Integral Rule (Interchanging differentiation and integration with respect to different variables.)

Suppose functions $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous on $[a, b] \times [c, d]$. Then

$$\frac{d}{dy} \left[\int_a^b f(x, y) dx \right] = \int_a^b \left[\frac{\partial}{\partial y} f(x, y) \right] dx.$$

Fourier Series

99. If $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{a}$ for $0 < x < a$, then $A_0 = \frac{1}{a} \int_0^a f(x) dx$ and $A_n = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a} dx$

100. If $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a}$ for $0 < x < a$, then $B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$

101. If $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$ for $-a < x < a$, then $a_0 = \frac{1}{2a} \int_{-a}^a f(x) dx$,

$$a_n = \frac{1}{a} \int_{-a}^a f(x) \cos \frac{n\pi x}{a} dx \text{ and } b_n = \frac{1}{a} \int_{-a}^a f(x) \sin \frac{n\pi x}{a} dx$$

102. If $f(x) = \sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi x}{2a}$ for $0 < x < a$, then $A_n = \frac{2}{a} \int_0^a f(x) \cos \frac{(2n-1)\pi x}{2a} dx$

103. If $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2a}$ for $0 < x < a$, then $B_n = \frac{2}{a} \int_0^a f(x) \sin \frac{(2n-1)\pi x}{2a} dx$

Generalized Fourier Series

104. Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

$$\text{If } f(r) = \sum_{m=1}^{\infty} a_m J_z\left(\frac{\alpha_m r}{a}\right) \text{ for } 0 < r < a, \text{ then } a_m = \frac{\int_0^a f(r) J_z\left(\frac{\alpha_m r}{a}\right) r dr}{\int_0^a J_z^2\left(\frac{\alpha_m r}{a}\right) r dr} = \frac{2 \int_0^a f(r) J_z\left(\frac{\alpha_m r}{a}\right) r dr}{a^2 J_{z+1}^2(\alpha_m)}$$

105. Suppose $0 < \beta_1 < \beta_2 < \dots$ are zeros of $J'_0(x)$.

$$\text{If } f(r) = a_0 + \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m r}{a}\right) \text{ for } 0 < r < a, \text{ then } a_0 = \frac{2}{a^2} \int_0^a f(r) r dr \text{ and}$$

$$a_m = \frac{\int_0^a f(r) J_0\left(\frac{\beta_m r}{a}\right) r dr}{\int_0^a J_0^2\left(\frac{\beta_m r}{a}\right) r dr} = \frac{2 \int_0^a f(r) J_0\left(\frac{\beta_m r}{a}\right) r dr}{a^2 J_0^2(\beta_m)}$$

106. Suppose $0 < \beta_1 < \beta_2 < \dots$ are zeros of $J'_z(x)$ and $z > 0$.

$$\text{If } f(r) = \sum_{m=1}^{\infty} a_m J_z\left(\frac{\alpha_m r}{a}\right) \text{ for } 0 < r < a, \text{ then}$$

$$a_m = \frac{\int_0^a f(r) J_z\left(\frac{\beta_m r}{a}\right) r dr}{\int_0^a J_z^2\left(\frac{\beta_m r}{a}\right) r dr} = \frac{2 \int_0^a f(r) J_z\left(\frac{\beta_m r}{a}\right) r dr}{a^2 [J_z^2(\beta_m) - J_{z-1}(\beta_m) J_{z+1}(\beta_m)]}$$

107. If $f(\phi) = \sum_{n=m}^{\infty} a_n P_n^m(\cos \phi)$ for $0 < \phi < \pi$ and $m > 0$, then

$$a_n = \frac{\int_0^{\pi} f(\phi) P_n^m(\cos \phi) \sin \phi d\phi}{\int_0^{\pi} [P_n^m(\cos \phi)]^2 \sin \phi d\phi} = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_0^{\pi} f(\phi) P_n^m(\cos \phi) \sin \phi d\phi$$

108. If $f(\phi) = \sum_{n=0}^{\infty} a_n P_n(\cos \phi)$ for $0 < \phi < \pi$, then

$$a_n = \frac{\int_0^{\pi} f(\phi) P_n(\cos \phi) \sin \phi d\phi}{\int_0^{\pi} [P_n(\cos \phi)]^2 \sin \phi d\phi} = \frac{2n+1}{2} \int_0^{\pi} f(\phi) P_n(\cos \phi) \sin \phi d\phi$$

109. If $f(\phi) = \sum_{k=0}^{\infty} a_k P_{2k}(\cos \phi)$ for $0 < \phi < \frac{\pi}{2}$, then

$$a_k = \frac{\int_0^{\frac{\pi}{2}} f(\phi) P_{2k}(\cos \phi) \sin \phi d\phi}{\int_0^{\frac{\pi}{2}} [P_{2k}(\cos \phi)]^2 \sin \phi d\phi} = (4k+1) \int_0^{\frac{\pi}{2}} f(\phi) P_{2k}(\cos \phi) \sin \phi d\phi$$

110. If $f(\phi) = \sum_{k=1}^{\infty} a_k P_{2k-1}(\cos \phi)$ for $0 < \phi < \frac{\pi}{2}$, then

$$a_k = \frac{\int_0^{\frac{\pi}{2}} f(\phi) P_{2k-1}(\cos \phi) \sin \phi d\phi}{\int_0^{\frac{\pi}{2}} [P_{2k-1}(\cos \phi)]^2 \sin \phi d\phi} = (4k-1) \int_0^{\frac{\pi}{2}} f(\phi) P_{2k-1}(\cos \phi) \sin \phi d\phi$$

Double Fourier Series

111. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$B_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} dy dx$$

112. If $f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$A_{00} = \frac{1}{ab} \int_0^a \int_0^b f(x, y) dy dx, \quad A_{n0} = \frac{2}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{n\pi x}{a} dy dx,$$

$$A_{0m} = \frac{2}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{m\pi y}{b} dy dx \quad \text{and} \quad A_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{b} dy dx$$

113. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{n0} = \frac{2}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} dy dx \quad \text{and} \quad C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} dy dx$$

114. If $f(x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D_{nm} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ for

$$(x, y) \in (0, a) \times (-b, b), \quad \text{then} \quad C_{0m} = \frac{1}{ab} \int_0^a \int_{-b}^b f(x, y) \sin \frac{m\pi x}{a} dy dx,$$

$$C_{nm} = \frac{2}{ab} \int_0^a \int_{-b}^b f(x, y) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} dy dx \quad \text{and} \quad D_{nm} = \frac{2}{ab} \int_0^a \int_{-b}^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx$$

115. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \cos \frac{(2n-1)\pi x}{2a} \cos \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{(2n-1)\pi x}{2a} \cos \frac{(2m-1)\pi y}{2b} dy dx$$

116. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b} dy dx$$

117. If $f(x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \cos \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{0m} = \frac{2}{ab} \int_0^b f(x, y) \cos \frac{(2m-1)\pi y}{2b} dy \text{ and } C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{n\pi x}{a} \cos \frac{(2m-1)\pi y}{2b} dy dx$$

118. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b} dy dx$$

119. If $f(x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \cos \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{0m} = \frac{2}{ab} \int_0^b f(x, y) \sin \frac{(2m-1)\pi y}{2b} dy \text{ and } C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \cos \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b} dy dx$$

120. If $f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \frac{(2n-1)\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{(2n-1)\pi x}{a} \sin \frac{(2m-1)\pi y}{2b} dy dx$$

Generalized Double Fourier Series

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

121. If $f(r, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} A_{mk} J_z\left(\frac{\alpha_m r}{a}\right) \cos \frac{k\pi\theta}{b}$ for $(r, \theta) \in (0, a) \times (0, b)$, then

$$A_{m0} = \frac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r, \theta) J_z\left(\frac{\alpha_m r}{a}\right) r d\theta dr \text{ and}$$

$$A_{mk} = \frac{4}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r, \theta) \cos \frac{k\pi\theta}{b} J_z\left(\frac{\alpha_m r}{a}\right) r d\theta dr$$

122. If $f(r, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} B_{mk} J_z\left(\frac{\alpha_m r}{a}\right) \sin \frac{k\pi\theta}{b}$ for $(r, \theta) \in (0, a) \times (0, b)$, then

$$B_{mk} = \frac{4}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r, \theta) \sin \frac{k\pi\theta}{b} J_z\left(\frac{\alpha_m r}{a}\right) r d\theta dr$$

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

123. If $f(r, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} A_{mk} J_z\left(\frac{\alpha_m r}{a}\right) \cos \frac{k\pi\theta}{b} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} B_{mk} J_z\left(\frac{\alpha_m r}{a}\right) \sin \frac{k\pi\theta}{b}$ for

$$(r, \theta) \in (0, a) \times (-b, b), \text{ then } A_{m0} = \frac{1}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_{-b}^b f(r, \theta) J_z\left(\frac{\alpha_m r}{a}\right) r \, d\theta \, dr,$$

$$A_{mk} = \frac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_{-b}^b f(r, \theta) \cos \frac{k\pi\theta}{b} J_z\left(\frac{\alpha_m r}{a}\right) r \, d\theta \, dr \text{ and}$$

$$B_{mk} = \frac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_{-b}^b f(r, \theta) \sin \frac{k\pi\theta}{b} J_z\left(\frac{\alpha_m r}{a}\right) r \, d\theta \, dr$$

124*. If $f(\theta, \phi) = \sum_{k=0}^{\infty} \sum_{n=m}^{\infty} A_{nk} P_n^m(\cos \phi) \cos \frac{k\pi\theta}{b}$ for $(\theta, \phi) \in (0, b) \times (0, \pi)$, then

$$A_{n0} = \frac{2n+1}{2b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_0^b f(\theta, \phi) P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi \text{ and}$$

$$A_{nk} = \frac{2n+1}{b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_0^b f(\theta, \phi) \cos \frac{k\pi\theta}{b} P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi$$

125*. If $f(\theta, \phi) = \sum_{k=0}^{\infty} \sum_{n=m}^{\infty} A_{nk} P_n^m(\cos \phi) \sin \frac{k\pi\theta}{b}$ for $(\theta, \phi) \in (0, b) \times (0, \pi)$, then

$$B_{nk} = \frac{2n+1}{b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_0^b f(\theta, \phi) \sin \frac{k\pi\theta}{b} P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi$$

126*. If $f(\theta, \phi) = \sum_{k=0}^{\infty} \sum_{n=m}^{\infty} A_{nk} P_n^m(\cos \phi) \cos \frac{k\pi\theta}{b} + \sum_{k=1}^{\infty} \sum_{n=m}^{\infty} B_{nk} P_n^m(\cos \phi) \sin \frac{k\pi\theta}{b}$ for

$$(\theta, \phi) \in (-b, b) \times (0, \pi), \text{ then } A_{n0} = \frac{2n+1}{4b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_{-b}^b f(\theta, \phi) P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi,$$

$$A_{nk} = \frac{2n+1}{2b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_{-b}^b f(\theta, \phi) \cos \frac{k\pi\theta}{b} P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi \text{ and}$$

$$B_{nk} = \frac{2n+1}{2b} \frac{(n-m)!}{(n+m)!} \int_0^\pi \int_{-b}^b f(\theta, \phi) \sin \frac{k\pi\theta}{b} P_n^m(\cos \phi) \sin \phi \, d\theta \, d\phi$$

* It also holds if $k = m$. For $m = 0$, $P_n^m(s) = P_n(s)$ and $\frac{(n-m)!}{(n+m)!} = 1$.

Generalized Triple Fourier Series

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

127**. If $f(\rho, \theta, \phi) = \sum_{q=0}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} A_{mkq} J_z\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \cos \frac{q\pi\theta}{b}$ for $(\rho, \theta, \phi) \in (0, a) \times (0, b) \times (0, \pi)$,

$$\text{then } A_{mk0} = \frac{2k+1}{a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_0^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi \, d\theta \, d\phi \, d\rho \text{ and}$$

$$A_{mkq} = \frac{2(2k+1)}{a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_0^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi \cos \frac{q\pi\theta}{b} \, d\theta \, d\phi \, d\rho$$

128**. If $f(\rho, \theta, \phi) = \sum_{q=1}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} B_{mkq} J_z\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \sin \frac{q\pi\theta}{b}$ for $(\rho, \theta, \phi) \in (0, a) \times (0, b) \times (0, \pi)$,

$$\text{then } B_{mkq} = \frac{2(2k+1)}{a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_0^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi \sin \frac{q\pi\theta}{b} \, d\theta \, d\phi \, d\rho$$

** For $l = 0$, $P_k^l(s) = P_k(s)$ and $\frac{(k-l)!}{(k+l)!} = 1$.

Suppose $0 < \alpha_1 < \alpha_2 < \dots$ are zeros of $J_z(x)$.

$$129^*. \text{ If } f(\rho, \theta, \phi) = \sum_{q=0}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} A_{mkq} J_z\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \cos \frac{q\pi\theta}{b} + \sum_{q=1}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} B_{mkq} J_z\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \sin \frac{q\pi\theta}{b}$$

for $(\rho, \theta, \phi) \in (0, a) \times (-b, b) \times (0, \pi)$, then

$$A_{mk0} = \frac{2k+1}{2a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi d\theta d\phi d\rho,$$

$$A_{mkq} = \frac{2k+1}{a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi \cos \frac{q\pi\theta}{b} d\theta d\phi d\rho \text{ and}$$

$$B_{mkq} = \frac{2k+1}{a^2 b J_{z+1}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_z\left(\frac{\alpha_m \rho}{a}\right) \rho P_k^l(\cos \phi) \sin \phi \sin \frac{q\pi\theta}{b} d\theta d\phi d\rho$$

$$130^*. \text{ If } f(\rho, \theta, \phi) = \sum_{q=0}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} A_{mkq} \rho^{-\frac{1}{2}} J_{z+\frac{1}{2}}\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \cos \frac{q\pi\theta}{a} +$$

$$\sum_{q=1}^{\infty} \sum_{k=l}^{\infty} \sum_{m=1}^{\infty} B_{mkq} \rho^{-\frac{1}{2}} J_{z+\frac{1}{2}}\left(\frac{\alpha_m \rho}{a}\right) P_k^l(\cos \phi) \sin \frac{q\pi\theta}{b}$$

for $(\rho, \theta, \phi) \in (0, a) \times (-b, b) \times (0, \pi)$, then

$$A_{mk0} = \frac{2k+1}{2a^2 b J_{z+\frac{3}{2}}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_{z+\frac{1}{2}}\left(\frac{\alpha_m \rho}{a}\right) \rho^{\frac{3}{2}} P_k^l(\cos \phi) \sin \phi d\theta d\phi d\rho,$$

$$A_{mkq} = \frac{2k+1}{a^2 b J_{z+\frac{3}{2}}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_{z+\frac{1}{2}}\left(\frac{\alpha_m \rho}{a}\right) \rho^{\frac{3}{2}} P_k^l(\cos \phi) \sin \phi \cos \frac{q\pi\theta}{b} d\theta d\phi d\rho \text{ and}$$

$$B_{mkq} = \frac{2k+1}{a^2 b J_{z+\frac{3}{2}}^2(\alpha_m)} \frac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho, \theta, \phi) J_{z+\frac{1}{2}}\left(\frac{\alpha_m \rho}{a}\right) \rho^{\frac{3}{2}} P_k^l(\cos \phi) \sin \phi \sin \frac{q\pi\theta}{b} d\theta d\phi d\rho$$

* For $l = 0$, $P_k^l(s) = P_k(s)$ and $\frac{(k-l)!}{(k+l)!} = 1$